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A New Derivation for the Field of a Time-Varying Charge in Einstein's Theory

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A fundamental problem in general relativity is the determination of the field produced by a source configuration consisting of a time-varying charge. By employing a generalized form for the electromagnetic energy-momentum tensor, it is possible to obtain an exact solution of the Einstein field equations for this distribution, without postulating a null fluid.

1. INTRODUCTION

In the following note, we arrive at a solution of the Einstein–Maxwell field equations for the field of a time-varying charge whose mass is constant. Our derivation obviates the necessity of Vaidya's hypothetical null fluid for the case of an arbitrary charge, $q(u)$.

The mixed forms of the Einstein field equations in nonempty space are written as

$$R_i^k - \tfrac{1}{2}\delta_i^k R = -8\pi T_i^k \tag{1.1}$$

where R_i^k and T_i^k are the usual components of the Ricci tensor and the energy-momentum tensor describing the curvature and energy content of space. For our derivation, external to the charge the energy-momentum tensor will only contain terms related to the null current and the electric field. We employ a general form for the energy-momentum tensor, which is

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derived in the Appendix:

$$T_i^k = -F^{km}F_{im} + \tfrac{1}{4}\delta_i^k F^{lm}F_{lm} - A_i J^k \tag{1.2}$$

where the electromagnetic field tensor is composed of the covariant curl of the four-potential

$$F_{ik} = A_{i;k} - A_{k;i} \tag{1.3}$$

and Maxwell's source equation is written in the usual manner,

$$F^{ik}{}_{;k} = J^i. \tag{1.4}$$

In the event that the components of J^i vanish external to a point, and that the charge and mass at that point are constant, these equations lead directly to the static Reissner–Nordstrom solution for the exterior field of a charged particle.

2. SOLUTION

We adopt the Bonnor–Vaidya metric using null polar (outgoing Eddington–Finkelstein) coordinates (Narlikar and Vaidya, 1947; Vaidya, 1950; Bonner and Vaidya, 1970, 1972; Patel and Shukla, 1974)

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2du\, dr + B\, du^2 \tag{2.1}$$

where we number the coordinates as follows:

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = \phi, \qquad x^4 = u$$

We specify the four-potential

$$A_i = r^{-1}q(u)\delta_i^4 \tag{2.2}$$

where $q(u)$ is an arbitrary function of the coordinate x^4. Then, from (1.3) we have

$$F_{14} = -F_{41} = r^{-2}q(u) \tag{2.3}$$

$$F^{14} = -F^{41} = -r^{-2}q(u) \tag{2.4}$$

The source of the radial electric field is found directly from (1.4) using the definition

$$F^{ik}{}_{;k}=\frac{1}{(-g)^{1/2}}\frac{\partial\left[(-g)^{1/2}F^{ik}\right]}{\partial x^k} \tag{2.5}$$

The only nonvanishing component of the four-current is J^1, whence

$$J^i=-r^{-2}\dot{q}(u)\delta_1^i \tag{2.6}$$

From the above definitions and the metric (2.1), the nonvanishing components of the energy-momentum tensor T_i^k are

$$T_1^1=-T_2^2=-T_3^3=T_4^4=\frac{q^2(u)}{2r^4}$$

$$T_4^1=\frac{q(u)\dot{q}(u)}{r^3} \tag{2.7}$$

We note that the Laue scalar associated with (1.2) vanishes. From the line element (2.1), we obtain the nonzero components of the Ricci tensor R_i^k,

$$R_1^1=R_4^4=-\frac{1}{2}\left(B_{11}+\frac{2B_1}{r}\right)$$

$$R_2^2=R_3^3=r^{-2}(1-rB_1-B) \tag{2.8}$$

$$R_4^1=-r^{-1}B_4$$

where the subscripts 1 and 4 on B denote differentiation with respect to r and u, respectively.

Since the trace of (1.2) vanishes, we use (2.8) and set the curvature scalar R equal to zero:

$$R=-\left(B_{11}+\frac{2B_1}{r}\right)+2r^{-2}(1-rB_1-B)=0 \tag{2.9}$$

The general solution of (2.9) is

$$B=1-\frac{2m}{r}+\frac{h(u)}{r^2} \tag{2.10}$$

Substituting (2.7), (2.8), and (2.10) into (1.1), we arrive at an exact solution if we specify

$$h(u)=4\pi q^2(u) \tag{2.11}$$

No null fluids are present (or necessary) in our solution.

3. THE CLASSICAL BONNER–VAIDYA SOLUTION

A similar solution was obtained by Bonner and Vaidya using the Einstein–Maxwell theory with a null fluid present. The Einstein field equations are then

$$R_i^k - \tfrac{1}{2}\delta_i^k R = -8\pi\left(T_i^k + v^k v_i\right) \tag{3.1}$$

where v^i is an *ad hoc* null fluid vector, i.e., one which satisfies the relation

$$g_{ik} v^i v^k = 0 \tag{3.2}$$

Their energy-momentum tensor, T_i^k, is *defined* as

$$T_i^k = -F^{km}F_{im} + \tfrac{1}{4}\delta_i^k F^{mn}F_{mn} \tag{3.3}$$

Equation (2.9) then admits a general solution

$$B = 1 - \frac{2m(u)}{r} + \frac{h(u)}{r^2} \tag{3.4}$$

with $h(u)$ specified identically by (2.11) and

$$k^2 = (4\pi r^2)^{-1}\left[-\dot{m}(u) + \frac{4\pi q(u)\dot{q}(u)}{r}\right] \tag{3.5}$$

where

$$v^i = k\delta_1^i \tag{3.6}$$

(They include the possibility of a time-varying mass.) This implies that an electric null current must be accompanied by a neutral null fluid current. It is the null fluid which transports the energy, and not the electromagnetic field in their picture (Bonner and Vaidya, 1972).

4. CONCLUSION

A solution of the Maxwell–Einstein field equation for an arbitrarily varying charge exists without postulating the presence of a null fluid. This solution is made possible through the realization of an extra term in the energy-momentum tensor which may be considered as inherent in a covariant formulation of the Poynting–Heaviside theorem.

APPENDIX

The canonical form of the electromagnetic energy-momentum tensor obtained by the principle of least action is nonsymmetric. That is,

$$T_i^k = A_{m;i} F^{km} + \tfrac{1}{4}\delta_i^k F_{lm} F^{lm} \tag{A.1}$$

To achieve symmetry, the term $(A_i F^{km})_{;m}$ is subtracted (Landau and Lifshitz, 1965) and we have

$$T_i^k = A_{l;i} F^{kl} + \tfrac{1}{4}\delta_i^k F_{lm} F^{lm} - \left(A_i F^{kl}\right)_{;l} \tag{A.2}$$

In the absence of four-currents, $F^{km}{}_{;m}$ vanishes and, for the electromagnetic field, (A.1) takes the conventional form

$$T_i^k = -F_{im} F^{km} + \tfrac{1}{4}\delta_i^k F_{lm} F^{lm} \tag{A.3}$$

However, (1.4) leads *us* to the following general form for the symmetrical stress-energy-momentum tensor of (A.2):

$$T_i^k = -F_{im} F^{km} + \tfrac{1}{4}\delta_i^k F_{lm} F^{lm} - A_i J^k \tag{A.4}$$

(terms obtained from $A_i J^i$ have, of course, vanished in the Lagrangian, $\Lambda = \frac{1}{4} F_{lm} F^{lm} + A_i J^i$ for our problem). The last term of (A.4) is consistent with the form of the Poynting–Heaviside vector intuitively postulated by Joseph Slepian in 1942.

Several comments must be made concerning the physical significance of equation (A.4). Since we are considering the source charge, $q(u)$, to be an arbitrary function of time we find ourselves in a position similar to that of Lyttleton and Bondi (1959). Our electromagnetic field equations are

Lorentz invariant but a physical condition

$$\nabla_k A^k = -\frac{q(u)}{r^2} \tag{A.5}$$

has replaced the conventional gauge condition

$$\nabla_k A^k = 0 \tag{A.6}$$

This same lack of gauge invariance is inherent in the original Bonner–Vaidya derivation.

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